Exterior derivative on $\mathbb{R}^n$

$$d\Big(\sum_J \omega_J dx^J\Big) = \sum_J d\omega_J \wedge dx^J = \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^J$$

eg $d(f) = \frac{\partial f}{\partial x^i} dx^i$

$$d(\omega_j dx^j) = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j$$

Properties

(a) $\mathbb{R}$-linear

(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(c) $d^2 = 0$

(d) $F^* d\omega = d F^* \omega$ for any smooth F. $U \to V$

{ in particular, take $\omega = x$ on $\mathbb{R}$, $dF = F^* dx = \sum \frac{\partial F}{\partial x^i} dx^i$

(e)

Pf (b) $d(u\, dx^I \wedge v\, dx^J)$

$= (u\,dv + v\,du)\, dx^I \wedge dx^J$

...

(c). 0-form by examples above

. $dd(u\,dx^I) = (dd u)\, dx^I = 0$ using (b)

(d) . true for fn's

. true for $\wedge$ $\quad F^* d(u\,dx^I) = F^*(du \wedge dx^I)$

. $d^2 = 0$ $\quad = d(u \circ F) \wedge \dots d(x^i \circ F), \dots$

$d(F^* u\, dx^I) = d$

Furthermore, (a),(b),(c),(e) uniquely determine d

Pf $d(u\,dx^I) \overset{(b),(c)}{=} du \wedge dx^I$

Thm For any M, $\exists!$ $d: \Lambda M \to \Lambda M$ satisfying (a),(b),(c),(e)

Pf (!) $d\theta$ is local: if $\theta = \hat{\theta}$ on U, then $d\theta = d\hat{\theta}$ on U

Pf $\eta(\theta - \hat{\theta}) = 0 \Rightarrow d(\quad) = 0$

$\eta(p) = 1$, $d\eta(p) = 0$

then ! on U determines $\theta(p)$.

($\exists$) Naturality $\Rightarrow$ can calculate in any chart

Stokes thm

By linearity + naturality, suffices to prove on H^n

$$\boxed{H^n: x^1 \geq 0}$$

For

$\omega = f\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1}$ (cpct supp)

$d\omega = \pm \frac{\partial f}{\partial x^i} dx^1 \wedge \ldots \wedge dx^n$

$\int_{H^n} d\omega = \ldots \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^i} dx^i\right) \ldots = 0$ Fubini

$\int_{\partial H^n} \omega = 0$ b/c $dx^1\big|_{\partial H^n} = 0$

For

$\omega = f\, dx^2 \wedge \ldots \wedge dx^{n+1}$

$d\left(f\, dx^2 \wedge \ldots \wedge dx^{n+1}\right)$

$\int_H \frac{\partial f}{\partial x^1} dx^1 \wedge \ldots \wedge dx^{n+1}$

$\int_{\mathbb{R}^{n+1}} d\omega = -\int_{\mathbb{R}^n_{2 \ldots n+1}} \omega$

$= \int_{-\mathbb{R}^n} \omega$

outward normal

- Lie derivative + Frobenius, reinterpreted
- Closed and exact forms, homotopy.

Frobenius

Lem $d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta[X,Y]$

⌜ For $\theta = u\,dv$

$LHS = (du \wedge dv)(X,Y)$

$du(X)\,dv(Y) - du(Y)\,dv(X)$

$= (Xu)(Yv) - (Yu)(Xv)$

$RHS = X(uYv) - Y(uXv) - u(XY - YX)v$ ⌟

Cor d and $[\,,\,]$ are dual:

If $X_1 \dots X_n$ is a frame, and $\theta^1 \dots \theta^n$ is the dual coframe, then

$$d\theta^i(X_j, X_k) = -\theta^i([X_j, X_k])$$

$\rightsquigarrow$ ① $D = \ker(\theta^1 \dots \theta^r)$ is integrable iff $I = \langle \theta^1 \dots \theta^r \rangle$ is a ~~differential~~ differential ideal

⌜ $Y^1 \dots Y^{n-r}$ dual frame; $\theta^i[Y_i, Y_j] = -d\theta^i(Y_i, Y_j)$

$= -\theta^k \wedge \varphi^l (Y_i, Y_j)$

$= 0$ ⌟

② A Lie algebra $\mathfrak{g}, [\,,\,]$ is equivalently an operator d_L on $\Lambda \mathfrak{g}^*$ satisfying (a), (b), (c).

Lie derivative

<u>Lie deriv for covariant</u>

$\mathcal{L}_V$ derivation of tensor algebra
s.t. contraction is <u>parallel</u>

$$\mathcal{L}_V(A\otimes B) = \mathcal{L}_V A \otimes B + A \otimes \mathcal{L}_V B$$
$$\mathcal{L}_V(A_i B^i) = (\mathcal{L}_V A)_i B^i + A_i (\mathcal{L}_V B)^i$$
$$V(\Theta(X)) = (\mathcal{L}_V\Theta)(X) + \Theta([V,X])$$
$$(\mathcal{L}_V\Theta)(X) = V(\Theta(X)) - \Theta([V,X])$$

<u>Lem</u> $\mathcal{L}_V\Theta := \frac{d}{dt}\Big|_{t=0} (\Phi_t)^*\Theta$ satisfies this

<u>For 1-forms of the form df</u>

$$\mathcal{L}_X df = \frac{d}{dt}\Big|_{t=0} \Phi_t^* df = d\left(\frac{d}{dt}\Big|_{t=0} \Phi_t^* f\right)$$
$$= d(Xf)$$

$$(\mathcal{L}_X df)(Y) = YXf$$
$$X(df\, Y) = XYf$$
$$df(\mathcal{L}_X Y) = (XY - YX)f$$

Interior product: $\iota_X\Theta(Y_1 \ldots Y_{k-1}) := \Theta(X, Y_1 \ldots Y_{k-1})$

Thm (Cartan)

$$\mathcal{L}_X = \iota_X d + d\iota_X$$

$\ulcorner$ • Both are derivations of degree 0

• they agree on fns + exact 1-forms $\lrcorner$